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THE DETERMINATION OF THE STRESS INTENSITY FACTOR IN A PLANE PROBLEM OF THE THEORY OF ELASTICITY*

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A method of determining the stress intensity factors (SIF) in plane problems of the theory of elasticity for regions with corner points is proposed. The case in which the corner points can be reached from infinity by a contour situated outside the region, was investigated in /1/.

When analysing the strength of a body it is sufficient (within the framework of the model of brittle fracture) to have available the asymptotic expression for the stresses in the neighbourhood of non-singular points of the boundary. Of course, the asymptotic form can be extracted from the solution of the boundary-value problem in the neighbourhood of such points, but it is clear that the construction of sufficiently reliable solutions in these regions will meet with considerable difficulties /2/. It is therefore important to obtain the asymptotic form without involving the direct solution of the boundary-value problem.

We know from the general theory of boundary-value problems for elliptic-type equations in regions with corner points /3/, that near the corner points the solution (as far as the problems of the theory of elasticity are concerned the discussion concerns the displacements) can be represented in the form of sums of an infinitely differentiable function and an asymptotic series, each term of which represents a solution of a homogeneous boundary-value problem for a wedge, with the same aperture angle as the corner point. Their multipliers are determined by the configuration of the region and the boundary conditions. The corresponding solutions for the plane problem of the theory of elasticity were obtained using polar coordinates /4/ and the apparatus of analytic functions /5/. Such solutions form a denumerable set. In problems with a physical content the solutions leading to unlimited energy are rejected. In problems of the theory of elasticity the most interesting solutions are those leading to unbounded stresses (there are, at most, two such solutions depending on the magnitude of the angle and form of the boundary conditions) and determining the SIF K_I and K_{II}.

In problems where the boundary conditions have the same form on both sides of the corner point, it can be asserted (provided that the polar angle is measured from the bisectrix) that one solution corresponds to a symmetrical distribution of the stresses (it gives the coefficient K_I), and the other to an antisymmetrical distribution (giving the coefficient K_{II}).

Let an elastic body occupy the region D with boundary S, containing at least one corner point, and let $2d$ be the angle between the semitangents in one of them (in which we require to find the SIF). Stresses $\mathbf{F}(q)$ are applied to the contour and we denote by $U(p)$ the solution required (the displacements are $\mathbf{U}(p)$ and the stresses are $\sigma(p)$).

Let us place the region D in the plane of the complex variable in such a manner that the corner point coincides with the zero, and the bisectrix of the angle coincides with the positive part of the real axis. This implies that the Kolosov-Muskhelishvili potentials $\varphi(z)$ and $\psi(z)$, corresponding to the solution $U(p)$ will have the following asymptotic form near the zero:

$$\varphi(z) \sim Cz^{\lambda}, \quad \psi(z) \sim ACz^{\lambda} \tag{1}$$

and the constants λ and A will be given by the equations (for the symmetrical case)

$$\sin 2\lambda\alpha + \lambda \sin 2\alpha = 0, \quad A = -\cos 2\lambda\alpha - \lambda \cos 2\alpha \tag{2}$$

The constant C is expressed in terms of the coefficient K_I and is to be determined. As we said before, we must have the solution of Eq.(2) within the range $0 < \lambda < 1$. Let us assume that such a solution exists, and denote it by λ_1 and A_1.

The formulas for the asymptotic forms of the displacements and stresses of the solution required $U(p)$ are, in polar coordinates, as follows:

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$$2\mu(U_r + iU_\theta) = C\rho^\lambda [\varkappa e^{i(\lambda-1)\theta} - \lambda e^{-i(\lambda-1)\theta} - Ae^{i(\lambda+1)\theta}] \tag{3}$$
$$\sigma_r + \sigma_\theta = 4C\rho^{\lambda-1}\,\mathrm{Re}\,[\lambda e^{i(\lambda-1)\theta}]$$
$$\sigma_\theta - \sigma_r + 2i\tau_{r\theta} = 2C\rho^{\lambda-1}[\lambda(\lambda-1)e^{i(\lambda-1)\theta} + \lambda Ae^{i(\lambda+1)\theta}]$$

Note that the constant $\varkappa = 3-4\nu$ for plane deformation and $x = 3-\nu/(1+\nu)$ for the plane state of stress (ν is Poisson's ratio). The above formulas establish the relation connecting the constant C with the coefficient K_{I} (K_{I} is the multiplier accompanying the component σ_θ) for $\theta = 0$.

In order to determine the SIF, it was suggested in /6-8/ that the solution with unlimited energy should be used, which we excluded from our discussion. Let us assume that we require to determine the SIF for the asymptotic expression with index λ_1. The structure of Eq.(2) implies that the value $-\lambda_1$ will also be a solution, and here the value of A will change (we shall denote it by A_{-1}). Let us write the solution obtained in complex form

$$\varphi_{-1}(z) = z^{-\lambda_1}, \quad \psi_{-1}(z) = A_{-1}z^{-\lambda_1} \tag{4}$$

We shall denote by $U^*(p)$ any solution of the boundary-value problem of the theory of elasticity for the region D with the asymptotic form (4) (henceforth, we shall denote all quantities referring to this solution by an asterisk).

In order to simplify our arguments, we shall assume that parts of the contour adjacent to the corner point are sections of sgraight lines of arbitrarily short length. We draw a circle of sufficiently small radius ε with centre at zero (appearing on the rectilinear segments), and eliminate from region D the points lying inside this circle. The Somigliana identity is applicable to the solution $U(p)$ required and to the auxiliary solution $U^*(p)$ introduced here, in the remaining region. The corresponding integral along the arc of the circle has the form

$$\int_{-\alpha}^{\alpha} (\sigma_r u_r^* + \tau_{r\theta}u_\theta^* - \sigma_r^* u_r - \tau_{r\theta}^* u_\theta)\, r\, d\theta \tag{5}$$

We will represent the displacement and stresses using the asymptotic expressions (1), (3) and (4). The integrals of the products of asymptotic expressions are calculated in explicit form (every term is a product of homogeneous functions). The remaining integrals have the order of smallness of ε and therefore, passing to the limit in the Somigliana formula, we arrive at the required relation

$$CB = \int_S [\mathbf{U}^*(q)\,\mathbf{T}_n\mathbf{U}(q) - \mathbf{U}(q)\,\mathbf{T}_n\mathbf{U}^*(q)]\, ds \tag{6}$$

where B is the integral shown above (the factor C must be taken outside the integral), and $\mathbf{T}_n$ is the stress operator. We note that $\mathbf{T}_n U(q) = F(q)$ are given boundary conditions.

In /7, 8/ a formula of the form (6) was used and a solution expressed in terms of asymptotic forms only was used as the solution $U^*(q)$ (generally speaking, the above papers dealt with mixed problems which did not lead to splitting into the symmetric and non-symmetric asymptotic forms). From formula (6) it follows that we must have available the displacements for the initial boundary-value problem. Since the displacements can be determined with greater accuracy than the stresses, this implies that the proposed approach will be more effective than that involving the extraction of the SIF from the stress field in the neighbourhood of the corner point.

We note that the asymptotic form (4) can only be used in the case when the corner point can be reached from infinity along a contour situated completely outside the region D. The fact is, that functions of the form z^λ (where λ) is fractional) are not unique over the whole plane, and in order to find a single-valued branch it is necessary to construct a cut joining the zero point with the point at infinity.

The above constraint can be removed by using the solution of the problem of the theory of elasticity which has a singularity of the required type at one of its corner points. Such a solution has been constructed * and formula (6) was used to obtain solutions of some problems. The author also remarked that fairly high accuracy can also be achieved in the approximate solution of the initial boundary-value problem (at least compared with the results in /2/).

It was suggested in /6/ that the solution $U^*(p)$ should be chosen in a special manner, requiring that relation $T_n U^* = 0$ hold at the boundary. Then formula (6) will become simpler and take the form

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*Romanchak V.M. Analysis of the stress state in the plane elastic problem for regions with corner points, based on the complex Betti formula. Candidate Dissertation. Minsk, 1986.

$$CB = \int_S \mathbf{U}^*(q)\,\mathbf{F}(q)\,ds \tag{7}$$

In order to construct the solution $U^*(p)$, we must choose a particular solution $U^0(p)$ which has a given singularity at the corner point, and construct a compensating solution U^{**} ($U^* = U^0 + U^{**}$). We chose, as a particular solution, solution (4) multiplied by the truncation, i.e. the infinity differentiable function $f(r)$ equal to unity for $r < \varepsilon$ (ε is a certain small number) and equal to zero for $r > 2\varepsilon$. Then in order to construct the compensating solution we must solve the inhomogeneous boundary-value problem. Introduction of the truncation takes care of the problem of attaining the corner point at infinity.

The above method was used in /1/ to construct the SIF in the case when the corner point could be reached at infinity, and this made the introduction of truncation unnecessary.

Below we discuss the problem of realizing the method proposed in /6/ in the case when the corner point is not attainable at infinity and the solution proposed by Romanchak is used as a particular solution (see the footnote).

We choose any point a situated outside the region which can be connected with the zero point (the corner point) by a contour lying completely outside D. Regarding this contour as a cut in the complex plane, we form the solution of the problem of the theory of elasticity in terms of the following single-valued functions:

$$\varphi^0(z) = z^{-\lambda_1}(z-a)^{\lambda_1},\quad \psi^0(z) = A_{-1}z^{-\lambda_1}(z-a)^{\lambda_1} \tag{8}$$

The constants λ_1 and A_{-1} are found, as before, from Eq.(2). It should be noted that it is appropriate not to treat (8) as a solution for the outside of a crescent with tips at a and at zero, and to speak of a solution for a plane with a cut connecting these points. The choice of the point a can be fairly arbitrary, but its positioning near the boundary or near the zero point would result in computational complications in the course of constructing the compensating solution.

We will use (8) to determine, at the points of the boundary, the components of the stress vector t_j^{**} for the solution u^{**}. Thus we arrive at the second fundamental boundary-value problem of the theory of elasticity. It is best solved using the singular integral equation obtained directly of the form /9/

$$u_i^{**}(q) + \int_S F_{ij}(q,q')\,u_j^{**}(q')\,ds_{q'} = \int_S G_{ij}(q,q')\,t_j^{**}(q')\,ds_{q'},\quad i,j=1,2 \tag{9}$$

$$G_{ij} = \frac{1}{8\pi\mu(1-\nu)}\left[(3-4\nu)\delta_{ij}\ln r - \frac{\xi_i\xi_j}{r^2}\right]$$

$$F_{ij}(q,q') = -\frac{1}{4\pi(1-\nu)r^2}\left\{(1-2\nu)(n_j\xi_i - n_j\xi_j) + \left[(1-2\nu)\delta_{ij} + \frac{2\xi_i\xi_j}{r^2}\right](\xi_k - n_k)\right\}$$

$$\xi_1 = x - x',\ \xi_2 = y - y',\ r = |q - q'|$$

(n_j are direction cosines). The expressions for the kernels given here are for the case of plane deformation.

The choice of Eqs.(9) is governed by the fact that it is the displacements that represent the function required, which means that no additional computations are needed when turning to formula (7). Eq.(9) is solved by the method of successive approximations, and the algorithm for solving the singular integrals is simplified by converting them to improper integrals by means of regular representations /10/. The algorithm is realized in the form of a universal program written in Fortran, with the points of the boundary specified in a discrete manner (the cut is situated along the real axis).

In the case of singly connected regions the corner point can always be reached at infinity, therefore it is best to consider the above method for the multiconnected regions. The problem for a region extending to infinity can be studied using the passage to the limit when the external contour is a sufficient distance away, and the conditions for the Somigliana identity to be valid (the integral along the outer contour vanishes) should be monitored. If the principal force vector applied to internal contours is equal to zero, then solution (8) can be used and in the general case we must ensure that the auxiliary solution U^0 tends to zero at infinity. This can be achieved in various ways, for example by passing to modifications of representation (8) using the relation

$$\varphi^0(z) = z^{-\lambda_1}(z-a)^{\lambda_1} - 1,\quad \psi^0(z) = A_{-1}z^{-\lambda_1}(z-a)^{\lambda_1} - A_{-1}$$

or

$$\varphi^0(z) = z^{-\lambda_1}(z-a)^{\lambda_1-1},\quad \psi^0(z) = A_{-1}z^{-\lambda_1}(z-a)^{\lambda_1-1}$$

Determination of the SIF K_{II} requires that a non-symmetric asymptotic form be introduced. To do this we use the solution

$$\varphi^0(z) = iz^{-\lambda}(z-a)^\lambda - i,\ \psi^0(z) = iAz^{-\lambda}(z-a)^\lambda - Ai \tag{10}$$

with parameters λ and A given by the equations

$$\sin 2\lambda\alpha - \lambda \sin 2\alpha = 0,\ A = \cos 2\lambda\alpha - \lambda\cos 2\alpha \tag{11}$$

We will now consider a method of computing the integral (7). The displacements can be represented in the form of the sum of two displacements, one of which (U^0) is given in explicit form and tends to infinity at the zero point, and the other (U^{**}) is determined in the course of solving the integral equation and is bounded everywhere. Therefore, it is best to evaluate the integrals in (7) separately for each term. For the displacement U^{**} we can use the discretization introduced in the course of solving the integral equation. Three zones must be introduced for the displacement U^0. In the immediate neighbourhood of the corner point the computations must be carried out analytically, neglecting the change in the multipliers of the power function. Next we have a zone in which we introduce a fairly fine discretization. In the last zone we can use the discretization introduced in the course of solving the integral equation. If the integral is solved at once, then the displacements at additional points of the contour must be found by interpolation.

We shall illustrate the effectiveness of the proposed algorithm by considering the problem for the outside of a square. We assume that Poisson's ratio is equal to 0.3, and that the modulus of elasticity is unity. The diagonal of the square is equal to 2, and is positioned so that one end of it lies at zero, and the other at -2. The point a lies at -1 The computations were carried out for the plane stress state and for plane deformation. The angle $2\alpha = 1.5\pi$, and, in accordance with (2), we have $\lambda_1 = 0.5445,\ A_1 = 0.8388$. The integral $B = 19.75$ for plane deformation, and 21.52 for the plane state of stress.

Initially we considered the problems for which the SIF were known. One of these solutions is given by the function

$$\varphi(z) = z^{\lambda_1}(z+1)^{-\lambda_1},\quad \varphi(z) = Az^{\lambda_1}(z+1)^{-\lambda_1} \tag{12}$$

Then the coefficient $C = 1$. Calculations the right-hand side of Eq.(7) gave values of $C = 0.9919$ for plane deformation and $C = 1.0005$ for the plane state of stress. Computations were also carried out for the problem in which the SIF is equal to zero. Such a solution is given, for example, by the functions $\varphi(z) = 0$ and $\psi(z) = (z+1)^{-1}$. The values of the coefficient C obtained were 0.0144, 0.0107 and 0.0092, respectively, when the contour was divided into 216, 260 and 320 segments.

In addition to the model problems discussed above, we carried out the computations for the case when a hydrostatic load (of unit magnitude) was applied to the contour, and in one case the load was specified over the whole contour, while in the other case the load was applied to the sides adjacent to the corner point. The following values were obtained for the coefficient K_I : 0.648 (0.640) and 0.613 (0.590), with the values in brackets referring to the plane state of stress.

Below we give the values of the displacement components U^* at points on two sides of the square (in the upper half-plane) for plane deformation and (in brackets) for the plane state of stress.

x	−2.0	−1.8	−1.15	−1.05	−1.0	−0.95
y	0	0.2	0.85	0.95	1.0	0.95
U_x^*	−0.579	−0.708	−0.997	−1.119	−1.282	−1.350
	(−0.752)	(−0.891)	(−1.209)	(−1.343)	(−1.520)	(−1.597)
U_y^*	0	0.508	1.474	1.725	1.925	2.075
	(0)	(0.560)	(1.621)	(1.896)	(2.114)	(2.287)
x	−0.7	−0.5	−0.2	−0.05	−0.01	−0.0033
y	0.7	0.5	0.2	0.05	0.01	0.0033
U_x^*	−1.515	−1.711	−2.466	−4.753	−10.86	−19.45
	(−1.773)	(−1.958)	(−2.818)	(−5.331)	(−12.04)	(21.46)
U_y^*	2.827	3.589	6.589	14.61	35.40	64.46
	(3.108)	(4.046)	(7.243)	(16.06)	(38.90)	(70.84)

For points in the lower half-plane the component U_x^* retains its value, but U_y^* changes its sign.

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A PSEUDOMACROCRACK IN AN ANISOTROPIC BODY*

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A pseudomacrocrack representing the usual crack in a composite material or in an inhomogeneous body is considered. The crack edges are pulled together by the non-disintegrated elements of the structure, and interact linearly. It is shown that in this case normal extension is sufficient for a non-zero value of the stress intensity factor K_{II} to occur at its tip. The problem is reduced to the Prandtl integro-differential vector equation for which an analytic solution is obtained. A relation is derived connecting the stress intensity factors K_I, K_{II}, with the stiffness of the joints between the edges, the elastic characteristics of the surrounding material and the external loads.

A classical example of a peseudomacrocrack is the macrocrack in a composite material consisting of a brittle ceramic matrix and tensile fibres pulling its edges together and preventing it from enlarging /1, 2/. A model of a pseudomacrocrack in an elastic, linearly anisotropic body in conditions of plane deformation was discussed in /3/. Here the relation between the stresses transmitted from the edge to the edge σ_{ni} and the opening of the edges w_j, was assumed to be linear:

$$\sigma_{ni}(\mathbf{x}) = k_{ij} w_j(\mathbf{x}) \tag{0.1}$$

where k_{ij} is a symmetric tensor. When the tensor k_{ij} is diagonal and the plane of elastic symmetry coincides with the plane of the pseudomacrocrack, the problem splits into two

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